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THE EFFECT OF COUPLE-STRESSES ON THE STRESS CONCENTRATION AROUND A CRACK*

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Abstract—A plane-strain solution is obtained, within the linearized couple-stress theory of elastic behavior, for the problem presented by a finite crack in a transverse field of uniform uni-axial tension. The singularities arising at the ends of the crack are studied in detail and the results, which are relevant to fracture considerations, are compared with their counterpart in classical elasto-statics.

INTRODUCTION

THE linearized couple-stress theory of elastic behavior, whose origins go back to the turn of the last century, has—for various and diverse reasons—attracted a renewed interest in recent years.† A particularly comprehensive study of this theory is due to Mindlin and Tiersten [2] (1962), while Mindlin [3] (1963) considered separately the corresponding two-dimensional theory of plane strain.

In [1] (1965) we applied the two-dimensional theory to several singular stress-concentration problems. The specific plane-strain problems treated there concern the disturbance produced by concentrated surface loads or discontinuously distributed shearing tractions applied to the boundary of a half-plane, as well as the geometrically induced concentration of stress arising at the corners of a smooth flat punch that is pressed against a semi-infinite elastic solid.

In the present paper, which is closely allied in scope to [1], we deal with yet another singular plane-strain problem: the stress concentration due to a transverse crack of finite length in an all-around infinite body that is otherwise in a state of uniform uni-axial tension at right angles to the plane of the crack. This problem, which is evidently of greater physical interest than those studied in [1], is also considerably more involved.

The motivation of this investigation is the same as that of [1]: to explore the implications of the couple-stress theory in circumstances for which the classical theory predicts unbounded concentrations of stress—which are accompanied by locally infinite deformation gradients. Our purpose, as before, is to ascertain the extent to which such pathological predictions are altered by the modified theory, which assigns an explicit role to the gradients of the rotation field in its constitutive law. This question is encouraged by the results of [2, 3] regarding the decrease, in the departure from the classical theory, of the stress concentration at a circular hole in a uni-axial field of stress. Nor can the present singular issue be safely dismissed on the grounds that "pathological questions are bound

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† See [1] for references to the related literature.

to yield pathological answers". For, the degree of pathology of a question depends upon the theoretical framework within which the answer is sought.

In Section 1 we summarize in compact notation, and extend to mixed boundary-value problems, the relevant theory of plane strain needed in the subsequent analysis. Here we also establish an elementary scheme for deducing from a known solution of a classical plane-strain problem the solution to an associated—though usually artificial—problem in the couple-stress theory. The crack problem to be considered is formulated in Section 2 and is reduced there, by means of an auxiliary half-plane problem, to a simultaneous system of dual integral equations. This system, in turn, is reduced in Section 3 to a one-dimensional integral equation of Fredholm's second kind that is amenable to a numerical treatment.

In Section 4 we study the limit behavior, in the transition to the classical theory, of the solution obtained in Section 3. Further, we determine in closed form the singularities at the ends of the crack arising in the couple-stress theory and compare these with their classical counterparts. We then present quantitative results based on the numerical solution of the Fredholm equation arrived at in Section 3. Finally, at the end of Section 4, we employ the principle of association discussed in Section 1 to generate an elementary "pseudo-solution" of the singular stress-concentration problem treated in this paper.

1. PLANE STRAIN IN THE LINEARIZED COUPLE-STRESS THEORY

We recall here the main features of the equilibrium theory of plane strain within the linearized couple-stress theory of homogeneous, centro-symmetric and isotropic, elastic solids. Most of the results about to be cited were first deduced by Mindlin [3] on the basis of purely two-dimensional considerations. These results were reaffirmed—through an appropriate specialization of the three-dimensional theory—in [1], where their connection with the three-dimensional plane problem is discussed in detail.

Throughout this section we employ the usual indicial notation with the understanding that Latin and Greek subscripts have the respective ranges (1, 2, 3) and (1, 2). Further, adhering to the symbolism adopted in [1], we call $\mathbf{u}$ and $\boldsymbol{\omega}$ the displacement and rotation vector fields, denote by $\mathbf{e}$ and $\boldsymbol{\kappa}$, the strain and curvature-twist tensor fields, while designating by $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ the tensor fields of stress and couple-stress.

Suppose now the medium under consideration occupies a cylindrical or prismatic region of space R and call D, with the boundary C, the open cross-section of R. Further, choose cartesian coordinates x_i such that the x_3-axis is parallel to the generators of R (Fig. 1). We assume that the body is in a state of plane deformations parallel to the plane $x_3 = 0$, so that

$$u_{\alpha,3} = 0, \qquad u_3 = 0 \quad \text{on} \quad \mathrm{R} \tag{1.1}$$

and adopt the normalization

$$\sigma_{kk} = 0 \quad \text{on} \quad \mathrm{R} \tag{1.2}$$

of the couple-stress field.† Upon entering the relevant system of fundamental field

† Recall from [1,2] that the skew-symmetric part of $\boldsymbol{\tau}$ and the isotropic part of $\boldsymbol{\sigma}$ remain indeterminate in the linearized couple-stress theory. Condition (1.2) serves to remove this indeterminacy and, as shown in [1], assures the continuous transition from the modified to the classical theory.

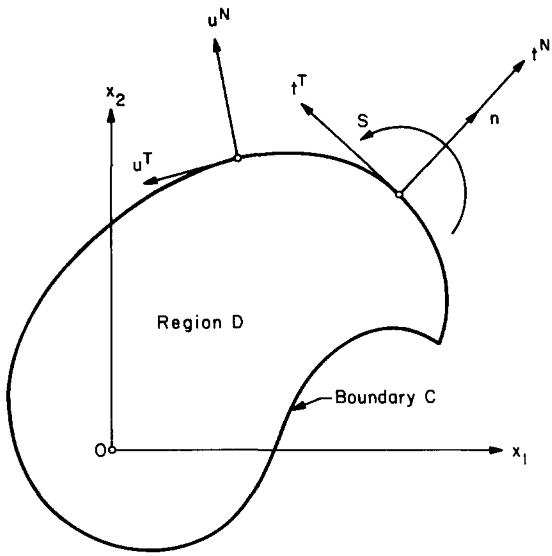

FIG. 1. Cross-section of body.

equations with (1.1), (1.2) one finds that ω_i, e_{ij}, $\varkappa_{ij}$, τ_{ij}, and σ_{ij} are independent of x_3 and that throughout D,

$$\left.\begin{aligned} &\omega_\alpha = e_{3i} = e_{i3} = \varkappa_{3i} = \varkappa_{\alpha\beta} = 0, \\ &\tau_{3\alpha} = \tau_{\alpha 3} = 0, \qquad \sigma_{33} = 0, \qquad \sigma_{\alpha\beta} = 0 \qquad (\alpha \neq \beta). \end{aligned}\right\} \tag{1.3}$$

To simplify the appearance of the resulting two-dimensional field equations, we introduce the abridged symbols

$$\omega = \omega_3, \qquad \varkappa_\alpha = \varkappa_{\alpha 3}, \qquad \sigma_\alpha = \sigma_{\alpha 3}, \tag{1.4}$$

write $\varepsilon_{\alpha\beta}$ for the components of the two-dimensional alternator, i.e.

$$\varepsilon_{12} = 1, \qquad \varepsilon_{21} = -1, \qquad \varepsilon_{11} = \varepsilon_{22} = 0, \tag{1.5}$$

and use the conventional notation for the symmetric component parts of a second-order tensor. The governing *kinematic relations* then become

$$\omega = \tfrac{1}{2}\varepsilon_{\alpha\beta}u_{\beta,\alpha}, \qquad e_{\alpha\beta} = u_{(\alpha,\beta)}, \qquad \varkappa_\alpha = \omega_{,\alpha}. \tag{1.6}$$

The *constitutive relations* furnish

$$e_{\alpha\beta} = \frac{1}{2\mu}[\tau_{(\alpha\beta)} - \nu\delta_{\alpha\beta}\tau_{\gamma\gamma}], \qquad \varkappa_\alpha = \frac{1}{4\mu l^2}\sigma_\alpha, \tag{1.7}$$

where μ, ν, and l, in this order, stand for the shear modulus, Poisson's ratio, and the characteristic length parameter of the material at hand, whereas $\delta_{\alpha\beta}$ is the Kronecker-delta. On the other hand, the *stress equations of equilibrium* in the present circumstances reduce to

$$\tau_{\beta\alpha,\beta} = 0, \qquad \varepsilon_{\alpha\beta}\tau_{\alpha\beta} + \sigma_{\alpha,\alpha} = 0, \tag{1.8}$$

provided the body-force and body-couple fields vanish identically, which we assume to be the case.

Equations (1.6), (1.7), (1.8), which must hold on D, are to be accompanied by appropriate boundary conditions. To this end let u^N, t^N and u^T, t^T be the scalar normal and tangential components of the displacement and traction vector on C and call s the axial component of the couple-traction vector on C (Fig. 1). Thus,

$$\left.\begin{array}{lll} u^N = u_\alpha n_\alpha, & u^T = \varepsilon_{\alpha\beta} u_\beta n_\alpha, & \\ t^N = \tau_{\beta\alpha} n_\alpha n_\beta, & t^T = \varepsilon_{\alpha\beta} \tau_{\gamma\beta} n_\alpha n_\gamma, & s = \sigma_\alpha n_\alpha, \end{array}\right\} \tag{1.9}$$

where n_α stands for the components of the unit outward normal of C. The *boundary conditions* then take the form

$$\left.\begin{array}{l} u^N = \overset{*}{u}{}^N \quad \text{or} \quad t^N = \overset{*}{t}{}^N, \qquad u^T = \overset{*}{u}{}^T \quad \text{or} \quad t^T = \overset{*}{t}{}^T, \\ \omega = \overset{*}{\omega} \quad \text{or} \quad s = \overset{*}{s} \quad \text{on} \quad C, \end{array}\right\} \tag{1.10}$$

in which letters carrying an asterisk represent given boundary values. Equations (1.10) are to convey the prescription at each regular point of C of one—though not necessarily the same—factor in each of the three products of surface quantities $u^N t^N$, $u^T t^T$, and ωs. The foregoing boundary conditions accordingly encompass mixed, as well as mixed-mixed, boundary-value problems.†

Equations (1.6), (1.7), (1.8) constitute fifteen equations in the fifteen unknowns u_α, ω, $e_{\alpha\beta}$, $\varkappa_\alpha$, $\tau_{\alpha\beta}$, and σ_α, which are to be determined subject to (1.10). As is apparent from the two-dimensional equilibrium analogue of the uniqueness theorem established by Mindlin and Tiersten [2], the solution of this boundary-value problem is unique (except possibly for an additive plane rigid displacement of D), provided the strain-energy density is positive definite. The latter is in the present instance found to be representable by the quadratic form

$$W(\boldsymbol{\tau}, \boldsymbol{\sigma}) = \frac{1}{4\mu}\left[\tau_{(\alpha\beta)}\tau_{(\alpha\beta)} - \nu\tau_{\alpha\alpha}\tau_{\beta\beta} + \frac{1}{2l^2}\sigma_\alpha\sigma_\alpha\right], \tag{1.11}$$

which is positive definite if and only if

$$\mu > 0, \qquad -1 < \nu < \tfrac{1}{2}, \qquad l \neq 0. \tag{1.12}$$

For future purposes we emphasize that the validity of the uniqueness theorem just cited rests upon appropriate regularity assumptions concerning the nature of D, the smoothness of the fields involved, and their behavior at infinity in the event that D is unbounded.

The complete (three-dimensional) plane-strain solution associated with the space-region R and the lateral boundary conditions (1.10) consists of the solution to the preceding subsidiary two-dimensional boundary-value problem, the last of (1.1), and (1.3), which is to be supplemented by

$$\tau_{33} = \nu\tau_{\alpha\alpha}, \qquad \sigma_{3\alpha} = 4\eta'\omega_{,\alpha}. \tag{1.13}$$

† In [1] we deduced only the boundary conditions for the *second* problem, in which t and s are assigned on C. The more general conditions (1.10) follow in the same manner from the corresponding specialization of (5.22) in [2].

Here η' is a second new elastic constant arising in the modified theory (see [2, 1]).

The field equations (1.6), (1.7), (1.8) imply the *displacement equations of equilibrium*

$$l^2\varepsilon_{\alpha\beta}\varepsilon_{\gamma\rho}\nabla^2 u_{\rho,\beta\gamma}+\nabla^2 u_\alpha+\frac{1}{1-2\nu}u_{\beta,\beta\alpha}=0, \tag{1.14†}$$

as well as the *stress equations of compatibility*

$$\varepsilon_{\alpha\beta}\sigma_{\alpha,\beta}=0, \qquad \sigma_\alpha=2l^2[\varepsilon_{\beta\gamma}\tau_{(\alpha\gamma),\beta}+\nu\varepsilon_{\alpha\beta}\tau_{\gamma\gamma,\beta}]. \tag{1.15‡}$$

Conversely, (1.14), (1.6), (1.7) imply (1.8), while (1.15), (1.7), (1.8) assure the existence of single-valued displacements u_α such that (1.6) hold true on D—provided D is simply connected.

Next, we recall Mindlin's [3] generalization of the Airy stress function in the classical theory of plane strain. The complete solution of (1.8) and the first of (1.15) admits the representation

$$\tau_{\alpha\beta}=\varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi_{,\gamma\rho}+\varepsilon_{\gamma\alpha}\psi_{,\beta\gamma}, \qquad \sigma_\alpha=\psi_{,\alpha}, \tag{1.16}$$

in terms of arbitrary (sufficiently smooth) stress functions ϕ and ψ. Substitution from (1.16) into the second of (1.15) yields the compatibility relations

$$(l^2\nabla^2\psi-\psi)_{,\alpha}=2(1-\nu)l^2\varepsilon_{\alpha\beta}\nabla^2\phi_{,\beta}, \tag{1.17}$$

from which, in turn,

$$\nabla^4\phi=0, \qquad l^2\nabla^4\psi-\nabla^2\psi=0. \tag{1.18}$$

Finally, from (1.16), (1.6), (1.7) follow

$$u_{(\alpha,\beta)}=\frac{1}{2\mu}[\varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi_{,\gamma\rho}-\nu\delta_{\alpha\beta}\nabla^2\phi+\varepsilon_{\gamma(\alpha}\psi_{,\beta)\gamma}], \tag{1.19}$$

$$\omega_{,\alpha}=\frac{1}{4\mu l^2}\psi_{,\alpha}. \tag{1.20}$$

Therefore, if D is simply connected, the problem under consideration is reducible to the determination of stress functions ϕ and ψ that satisfy (1.17) on D, such that the stresses and couple-stresses (1.16), as well as the displacements and the rotation obtained by integration of (1.19), (1.20), obey the boundary conditions (1.10). Alternatively the problem may be attacked with the aid of the complete solution of (1.14) in terms of generalized Papkovich–Neuber displacement potentials.§

In the presence of the normalization condition (1.2) $\sigma_{ij}\to 0$, $\tau_{ji}\to\tau_{ij}$ as $l\to 0$ and the modified equilibrium theory passes over into classical elastostatics. In particular, one recovers from (1.6), (1.7), (1.8), in this limit, the conventional field equations of plane strain

$$e_{\alpha\beta}=u_{(\alpha,\beta)}, \qquad e_{\alpha\beta}=\frac{1}{2\mu}[\tau_{\alpha\beta}-\nu\delta_{\alpha\beta}\tau_{\gamma\gamma}], \qquad \tau_{\alpha\beta,\beta}=0. \tag{1.21}$$

† ∇^2 is the Laplacian operator.

‡ Note that (1.15) imply $\nu\nabla^2\tau_{\gamma\gamma}-\varepsilon_{\alpha\gamma}\varepsilon_{\beta\rho}\tau_{(\alpha\beta),\gamma\rho}=0$. Conversely, this compatibility equation together with the second of (1.15) implies the first of (1.15).

§ See Section 11 in [2].

We now turn to a connection between the classical and the modified theory of plane strain that enables one to generate on the basis of a known solution to a classical plane-strain problem the solution to an associated boundary-value problem of plane strain in the couple-stress theory. The principle of association to which we are alluding may be stated in the form of the following

THEOREM. *Let $u_\alpha, e_{\alpha\beta}, \tau_{\alpha\beta}$ on D satisfy the classical field equations of plane strain* (1.21) *and define $\omega, \varkappa_\alpha, \sigma_\alpha$ on D through*

$$\omega = \tfrac{1}{2}\varepsilon_{\alpha\beta}u_{\beta,\alpha}, \qquad \varkappa_\alpha = \omega_{,\alpha}, \qquad \sigma_\alpha = 4\mu l^2 \varkappa_\alpha. \tag{1.22}$$

Then $u_\alpha, \omega, e_{\alpha\beta}, \varkappa_\alpha, \tau_{\alpha\beta}, \sigma_\alpha$ satisfy the modified field equations of plane strain (1.6), (1.7), (1.8). *Further,*

$$s \equiv \sigma_\alpha n_\alpha = 4\mu l^2 \frac{\partial \omega}{\partial n} \quad on \quad C. \tag{1.23}$$

The truth of this assertion is immediately inferred by inspection of (1.6), (1.7), (1.8) and (1.21), since the latter imply

$$\tau_{\beta\alpha} = \tau_{\alpha\beta}, \qquad \nabla^2\omega = 0 \quad \text{on} \quad D. \tag{1.24}$$

It follows in particular from the preceding theorem that if $u_\alpha, e_{\alpha\beta}, \tau_{\alpha\beta}$ are the solution of the classical plane-strain problem for D governed by the boundary conditions

$$t_\alpha \equiv \tau_{\beta\alpha}n_\beta = \overset{*}{t}_\alpha \quad \text{on} \quad C \tag{1.25}$$

and it so happens that

$$\frac{\partial \omega}{\partial n} = 0 \quad \text{on} \quad C, \tag{1.26}$$

then this solution, augmented by (1.22), supplies a solution of the modified plane-strain equations that obeys the boundary conditions

$$t_\alpha = \overset{*}{t}_\alpha, \qquad s = 0 \quad \text{on} \quad C, \tag{1.27}$$

i.e. corresponds to the same ordinary boundary tractions and to vanishing couple-tractions. It should be emphasized that the class of *non-singular* classical plane-strain solutions for which (1.26) holds true is rather narrow. Indeed, in the absence of singularities, one gathers from (1.26) and the second of (1.24), by virtue of the uniqueness theorem for the two-dimensional Neumann problem, that $\omega =$ const. on D. A non-trivial example of this kind is furnished by the conventional *axisymmetric* plane-strain problem, whose displacement field is irrotational. In this instance the solution of the associated problem in the modified theory is identical with the classical solution, the couple-stress field being zero identically.†

Consider next a solution $u_\alpha, e_{\alpha\beta}, \tau_{\alpha\beta}$ of the conventional plane-strain equations (1.21) on D that satisfies (1.25), (1.26) except for a finite number of points on C where it is *singular*. Then ω need no longer be constant on D. This eventuality is illustrated by the well-known classical solution corresponding to the half-plane under a concentrated normal load.‡ Here the associated solution in the couple-stress theory—consisting of

† Cf. the discussion in [3] of the stress concentration around a circular hole in an isotropic field of stress.
‡ See, for example, [4], art. 35.

$u_\alpha, e_{\alpha\beta}, \tau_{\alpha\beta}$ supplemented by $\omega, \varkappa_\alpha, \sigma_\alpha$ of (1.22)—meets the boundary conditions (1.27) except for a singularity at the point of application of the load. The elementary singular solution of the modified field equations thus generated is however a "pseudo-solution" of the analogous concentrated load-problem in the couple-stress theory since it fails to coincide with the appropriate limit of the solution to the problem of the half-plane under distributed normal tractions (and vanishing couple-tractions) established in [1], which supplies the correct solution of the problem at hand. The existence of such a pseudo-solution does not contradict the uniqueness theorem cited earlier, since this theorem does not apply to the singular problem under consideration.

Additional examples of singular plane-strain problems in the couple-stress theory that admit physically irrelevant pseudo-solutions of the foregoing type include the problem of the half-plane subjected to a rigid frictionless punch, which was treated in [1], as well as the crack-problem that constitutes our present objective. A pseudo-solution of the latter problem is exhibited in Section 4.

2. THE CRACK-PROBLEM. REDUCTION OF PROBLEM TO A SYSTEM OF DUAL INTEGRAL EQUATIONS

We now state the plane-strain problem in the couple-stress theory presented by a traction-free finite crack in a transverse field of uniform uni-axial tension. To this end, let D be the complement of the straight-line segment $-a \leq x_1 \leq a$, $x_2 = 0$ with respect to the entire x_1, x_2-plane, $2a$ being the length of the crack (Fig. 2). We seek a solution in

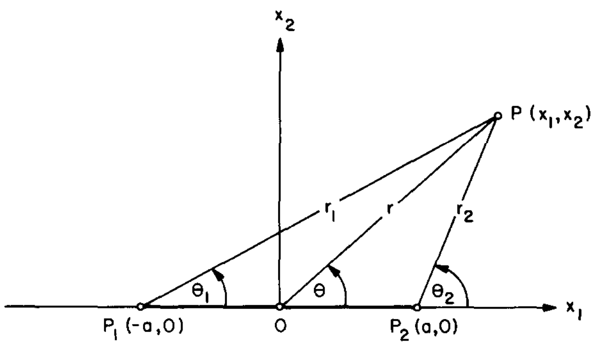

FIG. 2. Crack and coordinates.

D of the field equations (1.6), (1.7), (1.8) subject to the boundary conditions

$$\tau_{22}(x_1, 0) = 0, \qquad \tau_{21}(x_1, 0) = 0, \qquad \sigma_2(x_1, 0) = 0 \qquad (0 \leq |x_1| < a) \tag{2.1}$$

and the regularity conditions at infinity

$$\tau_{22} \to \tau_0, \qquad \tau_{11}, \tau_{12}, \tau_{21} \to 0, \qquad \sigma_\alpha \to 0 \quad \text{as} \quad r \to \infty, \tag{2.2}$$

where $r = \sqrt{(x_\alpha x_\alpha)}$ is the distance from the origin and the constant τ_0 denotes the intensity of the applied loading.

This boundary-value problem is reducible to the problem of a uniformly pressurized crack governed by the *inhomogeneous* boundary conditions

$$\tau_{22}(x_1, 0) = -\tau_0, \qquad \tau_{21}(x_1, 0) = 0, \qquad \sigma_2(x_1, 0) = 0 \qquad (0 \leq |x_1| < a) \tag{2.3}$$

together with the *homogeneous* conditions at infinity

$$\tau_{\alpha\beta} \to 0, \qquad \sigma_\alpha \to 0 \quad \text{as} \quad r \to \infty. \tag{2.4}$$

Indeed, if S and S' denote the solution of the problem characterized by (2.1), (2.2) and (2.3), (2.4), respectively, then

$$S = S' + \bar{S}, \tag{2.5}$$

where $\bar{S}$ is the solution in D of (1.6), (1.7), (1.8) appropriate to an undisturbed uniform field of uni-axial tension $\tau_{22} = \tau_0$ and is given by

$$\left.\begin{aligned} u_1(x_1, x_2) &= -\frac{\nu\tau_0}{2\mu}x_1, \qquad u_2(x_1, x_2) = \frac{(1-\nu)\tau_0}{2\mu}x_2, \qquad \omega = 0, \\ \tau_{22}(x_1, x_2) &= \tau_0, \qquad \tau_{11} = \tau_{12} = \tau_{21} = 0, \qquad \sigma_\alpha = 0. \end{aligned}\right\} \tag{2.6}$$

Further, in view of the symmetry about the x_1-axis of D and of the loading (2.3), (2.4), the solution to the problem of the pressurized crack must coincide on the upper half-plane D^+ $(-\infty < x_1 < \infty,\ 0 < x_2 < \infty)$ with the plane-strain solution for D^+ corresponding to the mixed-mixed boundary-value problem governed by (2.3), (2.4), and

$$u_2(x_1, 0) = 0, \qquad \omega(x_1, 0) = 0, \qquad \tau_{21}(x_1, 0) = 0 \qquad (a < |x_1| < \infty). \tag{2.7}$$

We observe that the boundary conditions (2.1), (2.3), and (2.7) are special cases of (1.10). The two crack problems introduced above are singular and their solutions accordingly remain indeterminate in the absence of additional information, concerning the nature of the singularities arising at the points $P_1(a, 0)$ and $P_2(-a, 0)$. We defer until later the supplementary specifications introduced in this connection.

In preparation for an attack upon the last mentioned problem, which is characterized by (2.3), (2.7), (2.4), we turn first to a more elementary mixed-mixed plane-strain problem for the upper half-plane D^+. This auxiliary problem corresponds to the boundary conditions

$$u_2(x_1, 0) = f_1(x_1), \qquad \omega(x_1, 0) = f_2(x_1), \qquad \tau_{21}(x_1, 0) = 0 \qquad (-\infty < x_1 < \infty) \tag{2.8}$$

together with conditions (2.4) at infinity. The prescribed functions f_α $(\alpha = 1, 2)$ appearing in (2.8) are assumed to be continuous and of bounded variation on $[-a, a]$; in addition they are required to obey

$$\left.\begin{aligned} &f_\alpha(x_1) = 0 \qquad (a < x < \infty), \\ &f_1(x_1) = f_1(-x_1), \qquad f_2(x_1) = -f_2(-x_1) \qquad (0 \leq |x_1| \leq a). \end{aligned}\right\} \tag{2.9}$$

The foregoing auxiliary problem may be solved on the basis of the generalized Airy-solution of the modified plane-strain equations recalled in Section 1, with the aid of the exponential Fourier transform. This method of solution consists in removing the x_1-dependence from the governing partial differential equations (1.16) to (1.20), as well as from the boundary and regularity conditions (2.8) and (2.4), by subjecting all of these equations to the Fourier transform with respect to x_1. In this manner (1.18), in the transform domain, give rise to a pair of ordinary fourth-order differential equations with constant coefficients, whose solution is immediate. The eight constants of integration thus emerging may subsequently be determined from the transforms of the two equations (1.17)† in conjunction with the transformed conditions (2.4), (2.8). Once the solution in the transform domain has been effected, the appropriate inversion theorem furnishes a complex integral representation for the solution in the physical domain, which is readily converted into real integral form.

Since the procedure just outlined is strictly analogous to that employed in [1] in connection with a similar boundary-value problem for the half-plane,‡ we may omit further details and cite directly the results obtained. For this purpose let

$$\left.\begin{aligned} q_1(s) &= \int_0^\infty f_1(x)\cos(sx)\,dx, \qquad q_2(s) = \int_0^\infty f_2(x)\sin(sx)\,dx, \\ q(s) &= sq_1(s)+q_2(s), \qquad \alpha(s) = \sqrt{(1+s^2)} \qquad (0 \le s < \infty), \end{aligned}\right\} \tag{2.10}$$

and write temporarily

$$\alpha \equiv \alpha(ls). \tag{2.11}$$

The solution of the auxiliary problem then appears as follows:

$$\left.\begin{aligned} u_1(x_1,x_2) &= \frac{1}{\pi}\int_0^\infty \left\{ \frac{sx_2-(1-2\nu)}{1-\nu} q_1(s)\exp(-sx_2) \right. \\ &\qquad \left. +4lq(s)[\alpha\exp(-\alpha x_2/l) - ls\exp(-sx_2)] \right\} \sin(sx_1)\,ds, \\ u_2(x_1,x_2) &= \frac{1}{\pi}\int_0^\infty \left\{ \left[2+\frac{sx_2}{1-\nu}\right] q_1(s)\exp(-sx_2) \right. \\ &\qquad \left. +4l^2 sq(s)[\exp(-\alpha x_2/l) - \exp(-sx_2)] \right\} \cos(sx_1)\,ds, \\ \omega(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty [q(s)\exp(-\alpha x_2/l) - sq_1(s)\exp(-sx_2)]\sin(sx_1)\,ds; \end{aligned}\right\} \tag{2.12§}$$

† Observe that while (1.17) implies (1.18), the converse is not true.

‡ In [1] the method of solution described here was applied to the plane-strain problem of the half-plane under given ordinary tractions and vanishing couple-tractions. Note that the role of x_1 and x_2 is reversed in [1] as compared to our present choice of coordinates.

§ There is no need to cite the corresponding formulas for the components of the strain and curvature-twist tensors.

$$\left.\begin{aligned}
\tau_{11}(x_1,x_2) &= -\frac{2\mu}{\pi}\int_0^\infty\left\{\frac{s}{1-\nu}(1-sx_2)q_1(s)\exp(-sx_2)\right.\\
&\qquad\left.-4lsq(s)[\alpha\exp(-\alpha x_2/l)-ls\exp(-sx_2)]\right\}\cos(sx_1)\,\mathrm{d}s,\\
\tau_{22}(x_1,x_2) &= -\frac{2\mu}{\pi}\int_0^\infty\left\{\frac{s}{1-\nu}(1+sx_2)q_1(s)\exp(-sx_2)\right.\\
&\qquad\left.+4lsq(s)[\alpha\exp(-\alpha x_2/l)-ls\exp(-sx_2)]\right\}\cos(sx_1)\,\mathrm{d}s,\\
\tau_{12}(x_1,x_2) &= -\frac{2\mu}{\pi}\int_0^\infty\left\{\frac{s^2x_2}{1-\nu}q_1(s)\exp(-sx_2)\right.\\
&\qquad\left.+4q(s)[\alpha^2\exp(-\alpha x_2/l)-l^2s^2\exp(-sx_2)]\right\}\sin(sx_1)\,\mathrm{d}s,\\
\tau_{21}(x_1,x_2) &= -\frac{2\mu}{\pi}\int_0^\infty\left\{\frac{s^2x_2}{1-\nu}q_1(s)\exp(-sx_2)\right.\\
&\qquad\left.+4l^2s^2q(s)[\exp(-\alpha x_2/l)-\exp(-sx_2)]\right\}\sin(sx_1)\,\mathrm{d}s,\\
\sigma_1(x_1,x_2) &= \frac{8\mu l^2}{\pi}\int_0^\infty\{sq_2(s)\exp(-\alpha x_2/l)\\
&\qquad+s^2q_1(s)[\exp(-\alpha x_2/l)-\exp(-sx_2)]\}\cos(sx_1)\,\mathrm{d}s,\\
\sigma_2(x_1,x_2) &= -\frac{8\mu l}{\pi}\int_0^\infty\{\alpha q_2(s)\exp(-\alpha x_2/l)\\
&\qquad+sq_1(s)[\alpha\exp(-\alpha x_2/l)-ls\exp(-sx_2)]\}\sin(sx_1)\,\mathrm{d}s.
\end{aligned}\right\}\quad(2.13)$$

It is easily verified that the integrals in (2.12), (2.13) are suitably convergent and that the solution given above indeed conforms to the field equations (1.6), (1.7), (1.8) on D^+, as well as to conditions (2.4), (2.8).

On comparing the boundary conditions (2.8) with (2.3), (2.7) one recognizes that the solution (2.12), (2.13) of the auxiliary problem coincides on D^+ with the desired solution of the pressurized-crack problem provided the "load-functions" f_α are determined so as to assure that the first and third of (2.3) hold true.

Now f_1 and f_2 enter (2.12), (2.13) only through their transforms q_1 and q_2, defined in (2.10). Further, from (2.10) and the inversion theorem for the Fourier cosine and sine transforms follows

$$f_1(x) = \frac{2}{\pi}\int_0^\infty q_1(s)\cos(sx)\,\mathrm{d}s,\qquad f_2(x) = \frac{2}{\pi}\int_0^\infty q_2(s)\sin(sx)\,\mathrm{d}s\qquad(0\le x<\infty).\qquad(2.14)$$

Hence q_1 and q_2 may hereafter be regarded as the basic unknowns. Applying the first of (2.3) to the second of (2.13), the third of (2.3) to the third of (2.12), and bearing in mind (2.14), the first of (2.9), as well as (2.10), we are led to the succeeding system of dual integral equations for q_α:

$$\left.\begin{aligned}&\int_0^\infty \left\{4ls[\alpha(ls)-ls][sq_1(s)+q_2(s)]+\frac{sq_1(s)}{1-\nu}\right\}\cos(sx)\,ds = \frac{\pi\tau_0}{2\mu},\\ &\int_0^\infty \{s[\alpha(ls)-ls]q_1(s)+\alpha(ls)q_2(s)\}\sin(sx)\,ds = 0 \qquad (0 \le x < a);\\ &\int_0^\infty q_1(s)\cos(sx)\,ds = 0, \qquad \int_0^\infty q_2(s)\sin(sx)\,ds = 0 \qquad (a < x < \infty).\end{aligned}\right\} \tag{2.15}$$

The treatment of (2.15) constitutes the burden of the remaining analysis.

3. FURTHER REDUCTION OF PROBLEM TO A FREDHOLM INTEGRAL EQUATION

Our immediate objective in dealing with (2.15) is to reduce this simultaneous system of dual integral equations to a single ordinary integral equation involving but one unknown function. To this end we replace x by x' in the first of (2.15), integrate with respect to x' over the range $0 \le x' \le x$ $(0 < x < a)$, and combine the resulting equation linearly with the second of (2.15) so as to eliminate $\alpha(ls)$. In this manner (2.15) is found to imply

$$\left.\begin{aligned}&\int_0^\infty \left[\frac{1}{1-\nu}q_1(s)-4l^2sq_2(s)\right]\sin(sx)\,ds = \frac{\pi\tau_0}{2\mu}x,\\ &\int_0^\infty \{s[\alpha(ls)-ls]q_1(s)+\alpha(ls)q_2(s)\}\sin(sx)\,ds = 0 \qquad (0 \le x < a);\\ &\int_0^\infty q_1(s)\cos(sx)\,ds = 0, \qquad \int_0^\infty q_2(s)\sin(sx)\,ds = 0 \qquad (a < x < \infty).\end{aligned}\right\} \tag{3.1}$$

Conversely, (3.1) is easily seen to imply (2.15), so that (3.1) and (2.15) are equivalent.

Next, define a function p through

$$p(s) = \frac{1}{1-\nu}q_1(s)-4l^2sq_2(s) \qquad (0 \le s < \infty). \tag{3.2}$$

From (3.2) one draws the identity

$$\left.\begin{aligned}\int_0^\infty \frac{p(s)}{s}\sin(sx)\,ds = \frac{1}{1-\nu}\int_0^x\int_0^\infty q_1(s)\cos(sx')\,ds\,dx'-4l^2\int_0^\infty q_2(s)\sin(sx)\,ds\\ (0 \le x < \infty),\end{aligned}\right\} \tag{3.3}$$

and (3.3), because of the last two of (3.1), yields

$$\int_0^\infty \frac{p(s)}{s}\sin(sx)\,ds = \frac{1}{1-\nu}\int_0^a\int_0^\infty q_1(s)\cos(sx')\,ds\,dx' \qquad (a < x < \infty). \tag{3.4}$$

Now use (3.2) and the first of (3.1) together with (3.4) and the first of (2.14) to confirm that p satisfies the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty p(s)\sin(sx)\,ds = \frac{\pi\tau_0}{2\mu}x \qquad (0 \leq x < a), \\ &\int_0^\infty \frac{p(s)}{s}\sin(sx)\,ds = \frac{\pi}{2(1-\nu)}\int_0^a f_1(x')\,dx' \qquad (a < x < \infty). \end{aligned}\right\} \tag{3.5}$$

Equations (3.5), in turn, are readily transformed into dual integral equations of a well-known type, whose solution is known.† In this manner one arrives at

$$p(s) = C_1 J_0(as) + \frac{\pi\tau_0 a}{2\mu}\frac{J_1(as)}{s} \qquad (0 < s < \infty), \tag{3.6}$$

where

$$C_1 = \frac{1}{1-\nu}\int_0^a f_1(x)\,dx - \frac{\pi\tau_0 a^2}{4\mu}, \tag{3.7}$$

while J_0, J_1 are the usual Bessel functions of the first kind. This solution may be verified directly by substitution into (3.5), with the aid of familiar Bessel integral-identities.‡ Elimination of q_2 from the second of (3.1) by means of (3.2) and (3.6) then yields the integral equation

$$\left.\begin{aligned} &\int_0^\infty \left\{ s[\alpha(ls) - ls] + \frac{\alpha(ls)}{4(1-\nu)l^2 s} \right\} q_1(s)\sin(sx)\,ds \\ &= \frac{C_1}{4l^2}\int_0^\infty \frac{\alpha(ls)}{s}J_0(as)\sin(sx)\,ds + \frac{\pi\tau_0 a}{8\mu l^2}\int_0^\infty \frac{\alpha(ls)}{s^2}J_1(as)\sin(sx)\,ds \qquad (0 \leq x < a), \end{aligned}\right\} \tag{3.8}$$

which involves q_1 as the only unknown function.§

Our next objective is the reduction of (3.8) to a Fredholm integral equation. For this purpose it is essential to restrict the nature of the displacement singularities admitted at the endpoints of the crack.‖ If $\mathring{u}$ denotes the displacement field of the classical solution to the pressurized-crack problem,¶ one has

$$\mathring{u}_2(x, 0) = \frac{(1-\nu)\tau_0}{\mu}\sqrt{(a^2 - x^2)} \qquad (0 \leq x \leq a). \tag{3.9}$$

† See Titchmarsh [5], p. 337 et seq.
‡ Watson [6], p. 405.
§ Note that the function f_1, which enters (3.8) through C_1, is linked to q_1 in accordance with the first of (2.14).
‖ Cf. the remarks concerning the uniqueness of the desired solution, following (2.7).
¶ See, for example, Sneddon [7], p. 425.

Guided by (3.9) and by the results in [1] regarding the modification of elastostatic singularities in the presence of couple-stresses, we now assume that $u_2(x,0)$ admits the representation

$$u_2(x,0) \equiv f_1(x) = C_2\sqrt{(a^2-x^2)}+\int_x^a \varphi(t)\sqrt{(t^2-x^2)}\,dt \qquad (0 \le x \le a), \tag{3.10}$$

where C_2 is a constant and φ a function continuous on $[0,a]$; moreover, both C_2 and φ are permitted to depend on the parameters a, ν, and l. It is clear from (3.10) and the continuity of φ, which will be confirmed *a posteriori*, that

$$u_{2,1}(x,0) = -\frac{C_2 x}{\sqrt{(a^2-x^2)}}+O(1) \quad \text{as} \quad x \to a \qquad (0 \le x < a) \tag{3.11}$$

so that (3.10) implies the preservation, in the departure from the classical theory, of the *order* of the singularities inherent in $u_{2,1}$.

In view of the identity

$$\int_0^a \sqrt{(a^2-x^2)}\cos(sx)\,dx = \frac{\pi a}{2s}J_1(as) \qquad (0 < s < \infty), \tag{3.12}†$$

we gather from (3.10), (2.10) that

$$q_1(s) = \frac{\pi}{2s}\left[C_2 a J_1(as)+\int_0^a t\varphi(t)J_1(ts)\,dt\right] \qquad (0 < s < \infty), \tag{3.13}$$

while (3.10) and the first of (3.7) furnish

$$C_1 = \frac{\pi}{4(1-\nu)}\left[a^2C_2+\int_0^a t^2\varphi(t)\,dt-\frac{(1-\nu)a^2\tau_0}{\mu}\right]. \tag{3.14}$$

At this point we note on the basis of the last of (2.10) that

$$\alpha(s) = s+\frac{1}{s+\alpha(s)} = s+\frac{1}{2s}-\frac{1}{2s[s+\alpha(s)]^2} \qquad (0 < s < \infty), \tag{3.15}$$

set for convenience

$$\beta(s) = s+\alpha(s), \qquad \gamma(s) = \frac{\alpha(s)-(1-2\nu)s}{[s+\alpha(s)]^2} \qquad (0 \le s < \infty), \tag{3.16}$$

and recall the identities‡

$$\int_0^\infty J_0(as)\sin(sx)\,ds = \begin{cases} 0 & (0 \le x < a) \\ \dfrac{1}{\sqrt{(x^2-a^2)}} & (a < x < \infty), \end{cases} \tag{3.17}$$

† Watson [6], p. 48.
‡ Watson [6], p. 405.

$$\int_0^\infty J_1(as)\sin(sx)\frac{ds}{s} = \begin{cases} \dfrac{x}{a} & (0 \le x \le a) \\[2ex] \dfrac{x}{a} - \dfrac{\sqrt{(x^2-a^2)}}{a} & (a \le x < \infty). \end{cases} \tag{3.18}$$

Substituting for $q_1(s)$ from (3.13) into (3.8), we obtain, after a permissible reversal of the order of the integrations with respect to s and t, upon using (3.15) to (3.18),

$$\begin{aligned} &C_2\left[(3-2\nu)x + \frac{a}{l}\int_0^\infty \frac{\gamma(ls)}{s^2}J_1(as)\sin(sx)\,ds\right] + (3-2\nu)x\int_0^a \varphi(t)\,dt \\ &\quad -(3-2\nu)\int_0^x \varphi(t)\sqrt{(x^2-t^2)}\,dt + \int_0^a t\varphi(t)\int_0^\infty \frac{\gamma(ls)}{ls^2}J_1(ts)\sin(sx)\,ds\,dt \\ &= \frac{2(1-\nu)C_1}{\pi l}\int_0^\infty \frac{J_0(as)}{s\beta(ls)}\sin(sx)\,ds + \frac{(1-\nu)\tau_0}{\mu}\left[x + \frac{a}{l}\int_0^\infty \frac{J_1(as)}{s^2\beta(ls)}\sin(sx)\,ds\right] \qquad (0 \le x \le a). \end{aligned} \tag{3.19}$$

Differentiating (3.19) with respect to x one arrives at

$$\begin{aligned} &C_2\left[3-2\nu + \frac{a}{l}\int_0^\infty \frac{\gamma(ls)}{s}J_1(as)\cos(sx)\,ds\right] + (3-2\nu)\int_0^a \varphi(t)\,dt \\ &\quad -(3-2\nu)x\int_0^x \frac{\varphi(t)}{\sqrt{(x^2-t^2)}}\,dt + \int_0^a t\varphi(t)\int_0^\infty \frac{\gamma(ls)}{ls}J_1(ts)\cos(sx)\,ds\,dt \\ &= \frac{2(1-\nu)C_1}{\pi l}\int_0^\infty \frac{J_0(as)}{\beta(ls)}\cos(sx)\,ds + \frac{(1-\nu)\tau_0}{\mu}\left[1 + \frac{a}{l}\int_0^\infty \frac{J_1(as)}{s\beta(ls)}\cos(sx)\,ds\right] \qquad (0 \le x \le a) \end{aligned} \tag{3.20}$$

and on passing to the limit in (3.20) as $x \to 0$, there results

$$\begin{aligned} &C_2\left[3-2\nu + \frac{a}{l}\int_0^\infty \frac{\gamma(ls)}{s}J_1(as)\,ds\right] + (3-2\nu)\int_0^a \varphi(t)\,dt + \int_0^a t\varphi(t)\int_0^\infty \frac{\gamma(ls)}{ls}J_1(ts)\,ds\,dt \\ &\qquad = \frac{2(1-\nu)C_1}{\pi l}\int_0^\infty \frac{J_0(as)}{\beta(ls)}\,ds + \frac{(1-\nu)\tau_0}{\mu}\left[1 + \frac{a}{l}\int_0^\infty \frac{J_1(as)}{s\beta(ls)}\,ds\right]. \end{aligned} \tag{3.21}$$

Subtracting (3.21) from (3.20) and setting

$$g(s) = \frac{1-\cos s}{s} \qquad (0 < s < \infty), \qquad g(0) = 0, \tag{3.22}$$

we reach

$$\begin{aligned} \int_0^x \frac{\varphi(t)}{\sqrt{(x^2-t^2)}}\,dt = -\frac{1}{3-2\nu}\Bigg\{&\frac{C_2 a}{l}\int_0^\infty \gamma(ls)g(sx)J_1(as)\,ds + \int_0^a t'\varphi(t')\int_0^\infty \frac{\gamma(ls)}{l}g(sx)J_1(t's)\,ds\,dt' \\ &-\frac{(1-\nu)a\tau_0}{\mu l}\int_0^\infty \frac{g(sx)}{\beta(ls)}J_1(as)\,ds - \frac{2(1-\nu)C_1}{\pi l}\int_0^\infty \frac{sg(sx)}{\beta(ls)}J_0(as)\,ds\Bigg\} \qquad (0 \le x \le a). \end{aligned} \tag{3.23}$$

The integral equation (3.23) has the familiar form

$$\int_0^x \frac{\varphi(t)}{\sqrt{(x^2-t^2)}}\,dt = f(x) \qquad (0 \le x \le a), \qquad f(0) = 0, \tag{3.24}$$

and (3.24) may be inverted† to give

$$\varphi(t) = \frac{2t}{\pi}\int_0^t \frac{1}{\sqrt{(t^2-x^2)}}\frac{d}{dx}[f(x)]\,dx \qquad (0 \le t \le a). \tag{3.25}$$

Also note that

$$\int_0^t \frac{1}{\sqrt{(t^2-x^2)}}\frac{d}{dx}[g(sx)]\,dx = \frac{\pi}{2t}J_1(ts), \tag{3.26}$$

as is easily verified by recourse to the known identities‡

$$\int_0^t \frac{\cos(yx)}{\sqrt{(t^2-x^2)}}\,dx = \frac{\pi}{2}J_0(yt), \qquad \int_0^s yJ_0(ty)\,dy = \frac{s}{t}J_1(ts). \tag{3.27}$$

Further, set

$$\xi = \frac{t}{a}, \qquad \eta = \frac{t'}{a} \qquad (0 \le t \le a, 0 \le t' \le a) \tag{3.28}$$

and adopt the dimensionless notation

$$\left.\begin{aligned} \lambda &= \frac{l}{a}, \qquad \Gamma_1 = \frac{4(1-\nu)\mu C_1}{\pi a^2 \tau_0}, \qquad \Gamma_2 = \frac{\mu C_2}{\tau_0}, \\ \Phi(\xi) &= \frac{a\mu}{\tau_0}(\sqrt{\xi})\varphi(a\xi) \qquad (0 \le \xi \le 1). \end{aligned}\right\} \tag{3.29}$$

Now apply the inversion formula (3.25) to (3.23), carry out the integrations with respect to t by means of (3.26), and then use (3.28), (3.29) to obtain

$$\Phi(\xi) + \int_0^1 \Phi(\eta)K(\xi,\eta)\,d\eta = \Gamma_\alpha F_\alpha(\xi) + F_3(\xi) \qquad (0 \le \xi \le 1). \tag{3.30§}$$

Here

$$K(\xi,\eta) = \frac{\sqrt{(\xi\eta)}}{(3-2\nu)\lambda}\int_0^\infty \gamma(\lambda s)J_1(\xi s)J_1(\eta s)\,ds \qquad (0 \le \xi \le 1, 0 \le \eta \le 1), \tag{3.31}$$

† See the derivation in [1] of (6.23) from (6.15).
‡ Watson [6], p. 18 and p. 48.
§ Recall that $\alpha = 1, 2$ and that summation over the repeated subscript is implied.

$$\left.\begin{aligned} F_1(\xi) &= -\frac{\sqrt{\xi}}{4(3-2\nu)\lambda^2}\int_0^\infty \frac{J_0(s)J_1(\xi s)}{[\beta(\lambda s)]^2}\,\mathrm{d}s, \qquad F_2(\xi) = -K(1,\xi),\\ F_3(\xi) &= \frac{(1-\nu)\sqrt{\xi}}{(3-2\nu)\lambda}\int_0^\infty \frac{1}{\beta(\lambda s)}J_1(s)J_1(\xi s)\,\mathrm{d}s \qquad (0\le\xi\le 1), \end{aligned}\right\} \tag{3.32†}$$

and β, γ are accounted for through (3.16), (2.10). On the other hand, (3.21) under the transformations (3.29) is carried into

$$\Gamma_2 G(1)+\int_0^1 \Phi(\eta)G(\eta)\frac{\mathrm{d}\eta}{\sqrt{\eta}} = k_1\Gamma_1+k_2, \tag{3.33}$$

where

$$G(\eta) = 1+\frac{\eta}{(3-2\nu)\lambda}\int_0^\infty \frac{\gamma(\lambda s)}{s}J_1(\eta s)\,\mathrm{d}s \qquad (0\le\eta\le 1), \tag{3.34}$$

with γ defined by (3.16), while the constants k_α are given by

$$k_1 = \frac{1}{2(3-2\nu)\lambda}\int_0^\infty \frac{J_0(s)}{\beta(\lambda s)}\,\mathrm{d}s, \qquad k_2 = \frac{1-\nu}{3-2\nu}\left[1+\frac{1}{\lambda}\int_0^\infty \frac{J_1(s)}{s\beta(\lambda s)}\,\mathrm{d}s\right]. \tag{3.35}$$

Finally, equation (3.14), in view of (3.29), becomes

$$\Gamma_1 = \Gamma_2+\int_0^1 \eta^{3/2}\Phi(\eta)\,\mathrm{d}\eta-(1-\nu). \tag{3.36}$$

To clarify matters we emphasize that K and F_i $(i = 1,2,3)$ in (3.30) and G in (3.33) are *known* functions, whose definitions are supplied by (3.31), (3.32), (3.34); similarly k_α $(\alpha = 1,2)$ in (3.33) are the *known* constants furnished by (3.35). In contrast, Γ_α $(\alpha = 1,2)$, which enter (3.30), (3.33), (3.36), are functionals of the *unknown* function Φ. As is at once apparent, the elimination of Γ_α from (3.30) by means of (3.33), (3.36) yields a single inhomogeneous integral equation of Fredholm's second kind for Φ. The kernel of the equation thus obtained is, however, asymmetric and unwieldy. For this reason we avoid the elimination just described and instead reduce the determination of Φ to the solution of three *independent* Fredholm equations, all of which have the common symmetric kernel K.

With this objective in mind consider the triplet of Fredholm equations

$$\Phi_i(\xi)+\int_0^1 \Phi_i(\eta)K(\xi,\eta)\,\mathrm{d}\eta = F_i(\xi) \qquad (0\le\xi\le 1), \qquad (i = 1,2,3), \tag{3.37}$$

where K and F_i are defined by (3.31) and (3.32), respectively. If $\lambda > 0$ and Poisson's ratio‡ obeys $0 \le \nu \le \frac{1}{2}$, then the symmetric kernel K is continuous on its square domain of definition $[0,1]\times[0,1]$, while F_i is continuous on $[0,1]$. Moreover, it is not difficult to show§ that K is positive definite. Consequently‖ (3.37) assure the existence of unique solutions Φ_i that are continuous on $[0,1]$.

† To render the definition of F_1 continuous on $[0,1]$ we have used (3.15), the first of (3.16), and the identity (9) on p. 406 of [6].

‡ Observe that ν enters K and F_i explicitly, as well as through γ and β.

§ Cf. the Appendix of [1].

‖ See, for example, Courant and Hilbert [8], p. 116.

Now set

$$\Phi(\xi) = \Gamma_1\Phi_1(\xi)+\Gamma_2\Phi_2(\xi)+\Phi_3(\xi) \qquad (0 \le \xi \le 1). \tag{3.38}$$

It is clear from the linearity of (3.30) and from (3.37) that Φ of (3.38) satisfies (3.30) for every choice of the constants Γ_α ($\alpha = 1,2$). It remains therefore merely to determine Γ_α consistent with (3.33) and (3.36). Indeed, substitution for Φ from (3.38) into (3.33), (3.36) leads to two simultaneous linear algebraic equations in Γ_1 and Γ_2, whose solution is given by

$$\left.\begin{aligned} \Gamma_1 &= \frac{[G(1)+A_2](1-\nu-B_3)-(k_2-A_3)(1+B_2)}{(k_1-A_1)(1+B_2)-[G(1)+A_2](1-B_1)}, \\ \Gamma_2 &= \frac{(k_1-A_1)(1-\nu-B_3)-(k_2-A_3)(1-B_1)}{(k_1-A_1)(1+B_2)-[G(1)+A_2](1-B_1)}, \end{aligned}\right\} \tag{3.39}$$

provided

$$A_i = \int_0^1 \Phi_i(\xi)G(\xi)\frac{d\xi}{\sqrt{\xi}}, \qquad B_i = \int_0^1 \xi^{3/2}\Phi_i(\xi)\,d\xi. \tag{3.40}$$

Equations (3.38), (3.39), (3.40) render Φ fully determinate, once the solutions Φ_i ($i = 1,2,3$) of the Fredholm equations (3.37) have been found. This task was accomplished on an electronic computer. It follows from (3.38) and the continuity of the functions Φ_i that Φ is continuous on $[0,1]$, whereas (3.37), (3.38), and (3.31), (3.32) insure that

$$\Phi(\xi) = O(\sqrt{\xi}) \quad \text{as} \quad \xi \to 0. \tag{3.41}$$

Hence φ, which is linked to Φ through the last of (3.29), is continuous on $[0,a]$. This conclusion enables one to justify *a posteriori* various formal manipulations used in the course of the preceding analysis.

From (3.29), (3.13), (3.6), and (3.2) one gathers that the solution of the original system of dual integral equations (2.15) admits the representation†

$$\left.\begin{aligned} q_1(s) &= \frac{a\pi\tau_0}{2\mu s}\left[\Gamma_2 J_1(as)+\int_0^1 (\sqrt{\xi})\Phi(\xi)J_1(as\xi)\,d\xi\right], \\ q_2(s) &= \frac{\pi\tau_0}{8(1-\nu)\mu\lambda^2 as^2}\left[(\Gamma_2-1+\nu)J_1(as)-\frac{as}{2}\Gamma_1 J_0(as)\right. \\ &\quad \left.+\int_0^1 (\sqrt{\xi})\Phi(\xi)J_1(as\xi)\,d\xi\right] \qquad (0 < s < \infty). \end{aligned}\right\} \tag{3.42}$$

Insertion of these values of $q_\alpha(s)$ ($\alpha = 1,2$) in (2.12), (2.13) furnishes the desired solution on the upper half-plane D^+ of the pressurized-crack problem governed by (2.3), (2.4). The solution corresponding to the problem of a crack in a transverse field of uniform tension, which is characterized by (2.1), (2.2), is then immediate from (2.5), (2.6).

The improper integral representations for the auxiliary functions K, F_i, G, and for the constants k_α, appearing in (3.31), (3.32), (3.34), and (3.35), are inconvenient for numerical purposes because of the infinite range of integration and the oscillatory character of the

† Observe that the steps leading from (2.15) to (3.30), (3.33), and (3.36) may be reversed.

integrands concerned. Alternative representations, which are free from these deficiences, are readily deduced by means of suitable contour integrations, the details of which are omitted here. The results obtained in this manner, which were employed in the numerical evaluations carried out, involve the modified Bessel functions I_1, K_0, K_1 and have the following form:

$$\left.\begin{aligned}
&K(\xi,\eta) = K(\eta,\xi) = \\
&\qquad \frac{2\sqrt{(\xi\eta)}}{(3-2\nu)\pi\lambda^2}\int_0^1 \sqrt{(1-t^2)}[1-4(1-\nu)t^2]I_1(\eta t/\lambda)K_1(\xi t/\lambda)\,dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le \eta \le \xi \le 1) \\
&F_1(\xi) = -\frac{\sqrt{\xi}}{(3-2\nu)\pi\lambda^3}\int_0^1 t\sqrt{(1-t^2)}I_1(\xi t/\lambda)K_0(t/\lambda)\,dt \qquad (0 \le \xi \le 1), \\
&F_3(\xi) = \frac{2(1-\nu)\sqrt{\xi}}{(3-2\nu)\pi\lambda^2}\int_0^1 \sqrt{(1-t^2)}I_1(\xi t/\lambda)K_1(t/\lambda)\,dt \qquad (0 \le \xi \le 1), \\
&G(\xi) = 1-\frac{2\xi}{(3-2\nu)\pi\lambda}\int_0^1 \frac{\sqrt{(1-t^2)}}{t}[1-4(1-\nu)t^2][K_1(\xi t/\lambda)-\lambda/\xi t]\,dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 < \xi \le 1),\ G(0) = 1, \\
&k_1 = \frac{1}{(3-2\nu)\pi\lambda^2}\int_0^1 \sqrt{(1-t^2)}K_0(t/\lambda)\,dt, \\
&k_2 = \frac{1-\nu}{3-2\nu}\left\{1-\frac{2}{\pi\lambda}\int_0^1 \frac{\sqrt{(1-t^2)}}{t}[K_1(t/\lambda)-\lambda/t]\,dt\right\}.
\end{aligned}\right\} \tag{3.43}$$

4. LIMIT CONSIDERATIONS AND NUMERICAL RESULTS. PSEUDO-SOLUTION OF THE CRACK PROBLEM

We examine next the behavior of the solution to the problem of the pressurized crack† in the limit as the characteristic length-ratio $\lambda = l/a$ tends to zero. Whereas we have so far, for the sake of brevity, suppressed the argument λ of various functions that depend on this parameter, it is helpful for our present purpose to make their λ-dependence explicitly apparent. Accordingly, if $f(s)$ are values of a function that depends on λ as well, we now write $f(s;\lambda)$ in place of $f(s)$.

From (3.42) and (3.30), (3.31), (3.32) follows (see the Appendix for a sketch of the required proof) for every s in $(0,\infty)$ that

$$q_1(s;\lambda) = \frac{a\pi(1-\nu)\tau_0}{2\mu s}J_1(as)+o(1), \qquad q_2(s;\lambda) = o(\lambda^{-2}) \quad \text{as} \quad \lambda \to 0. \tag{4.1}$$

Using (4.1) in conjunction with (2.12), (2.13), and passing to the limit as $\lambda \to 0$ under the integral signs—as is readily seen to be permissible—one arrives at integral representations

† Because of (2.5), (2.6), the solution appropriate to a crack in a transverse field of tension does not require separate attention.

for the limit fields $\mathring{u}_\alpha$, $\mathring{\omega}$, $\mathring{\tau}_{\alpha\beta}$, $\mathring{\sigma}_{\alpha\beta}$ defined by

$$\left.\begin{aligned} \mathring{u}_\alpha(x_1, x_2) &= \lim_{\lambda\to 0} u_\alpha(x_1, x_2; \lambda), &\quad \mathring{\omega}(x_1, x_2) &= \lim_{\lambda\to 0} \omega(x_1, x_2; \lambda), \\ \mathring{\tau}_{\alpha\beta}(x_1, x_2) &= \lim_{\lambda\to 0} \tau_{\alpha\beta}(x_1, x_2; \lambda), &\quad \mathring{\sigma}_\alpha(x_1, x_2) &= \lim_{\lambda\to 0} \sigma_\alpha(x_1, x_2; \lambda). \end{aligned}\right\} \tag{4.2}$$

The definite integrals thus arising may, with the aid of known Bessel integral-identities (Watson [6], p. 386), be evaluated in closed form in terms of elementary functions of the polar coordinates (r, θ), (r_1, θ_1), and (r_2, θ_2) indicated in Fig. 2. The computations just outlined lead to the results:

$$\left.\begin{aligned} \mathring{u}_1(x_1, x_2) &= \frac{\tau_0}{2\mu}\sqrt{(r_1 r_2)}\left\{(1-2\nu)\cos\left[\frac{1}{2}(\theta_1+\theta_2)\right] - (1-2\nu)\frac{r}{\sqrt{(r_1 r_2)}}\cos\theta \right. \\ &\quad \left. - \frac{r^2}{r_1 r_2}\sin\theta\sin\left[\theta - \frac{1}{2}(\theta_1+\theta_2)\right]\right\}, \\ \mathring{u}_2(x_1, x_2) &= \frac{\tau_0}{2\mu}\sqrt{(r_1 r_2)}\left\{2(1-\nu)\sin\left[\frac{1}{2}(\theta_1+\theta_2)\right] - (1-2\nu)\frac{r}{\sqrt{(r_1 r_2)}}\sin\theta \right. \\ &\quad \left. - \frac{r^2}{r_1 r_2}\sin\theta\cos\left[\theta - \frac{1}{2}(\theta_1+\theta_2)\right]\right\}, \\ \mathring{\omega}(x_1, x_2) &= \frac{\tau_0(1-\nu)r}{\mu\sqrt{(r_1 r_2)}}\sin\left[\theta - \frac{1}{2}(\theta_1+\theta_2)\right]; \\ \mathring{\tau}_{11}(x_1, x_2) &= \frac{\tau_0 r}{\sqrt{(r_1 r_2)}}\left\{\cos\left[\theta - \frac{1}{2}(\theta_1+\theta_2)\right] - \frac{a^2\sin\theta}{r_1 r_2}\sin\left[\frac{3}{2}(\theta_1+\theta_2)\right]\right\} - \tau_0 \\ \mathring{\tau}_{22}(x_1, x_2) &= \frac{\tau_0 r}{\sqrt{(r_1 r_2)}}\left\{\cos\left[\theta - \frac{1}{2}(\theta_1+\theta_2)\right] + \frac{a^2\sin\theta}{r_1 r_2}\sin\left[\frac{3}{2}(\theta_1+\theta_2)\right]\right\} - \tau_0 \\ \mathring{\tau}_{12}(x_1, x_2) &= \mathring{\tau}_{21}(x_1, x_2) = \frac{\tau_0 a^2 r}{(r_1 r_2)^{3/2}}\sin\theta\cos\left[\frac{3}{2}(\theta_1+\theta_2)\right]; \qquad \mathring{\sigma}_\alpha(x_1, x_2) = 0. \end{aligned}\right\} \tag{4.3}$$

Equations (4.3) are found to be valid for all (x_1, x_2) in the closure of D^+ with the exception of the points $x_1 = \pm a$, $x_2 = 0$.

The fields $\mathring{u}_\alpha$, $\mathring{\omega}$, and $\mathring{\tau}_{\alpha\beta}$ given by (4.3) are identical with the corresponding fields of displacement, rotation, and stress predicted by the classical solution† to the problem of the pressurized crack. Consequently, as $\lambda \to 0$, the solution based on the couple-stress theory passes over continuously into its classical counterpart.

Our chief interest in the solution to the crack problem deduced in Section 3 concerns its behavior at the (singular) endpoints of the crack. An examination of (2.12), (2.13), with q_α given by (3.42), reveals that all improper integrals involved in (2.12) are convergent throughout the closure of the half-plane D^+. In contrast, the integrals in (2.13) are found to be divergent at $x_1 = \pm a$, $x_2 = 0$. This divergence is evidently due to the behavior of the corresponding integrands as $s \to \infty$. With a view toward exhibiting the singular behavior of $\tau_{\alpha\beta}$ and σ_α at the roots of the crack we are therefore led to deduce, with the aid of (3.15), asymptotic expansions in s (valid as $s \to \infty$) of the integrands appearing in

† See, for example, Sneddon [7], p. 427. Note, however, that Sneddon's choice of coordinates differs from ours.

(2.13) and to identify the dominating terms in these expansions whose contributions to the stress and couple-stress fields become unbounded as $r_\gamma \to 0$. The foregoing contributions may then be determined in closed elementary form by means of familiar Bessel integral-identities (Watson [6], p. 386). This process† yields the following estimates, which hold true as $r_\gamma \to 0$ for every fixed *positive* λ:

$$\left.\begin{aligned}
&u_\alpha(x_1,x_2;\lambda) = O(1), \qquad \omega(x_1,x_2;\lambda) = O(1),\\
&\tau_{11}(x_1,x_2;\lambda) = -\frac{\tau_0(1-2\nu)}{1-\nu}\Gamma_2(\lambda)\frac{r}{\sqrt{(r_1r_2)}}\left\{\cos\left[\theta-\frac{1}{2}(\theta_1+\theta_2)\right]\right.\\
&\qquad\left.-\frac{a^2\sin\theta}{r_1r_2}\sin\left[\frac{3}{2}(\theta_1+\theta_2)\right]\right\}+O(1),\\
&\tau_{22}(x_1,x_2;\lambda) = \frac{\tau_0\Gamma_2(\lambda)}{1-\nu}\frac{r}{\sqrt{(r_1r_2)}}\left\{(3-2\nu)\cos\left[\theta-\frac{1}{2}(\theta_1+\theta_2)\right]\right.\\
&\qquad\left.-(1-2\nu)\frac{a^2\sin\theta}{r_1r_2}\sin\left[\frac{3}{2}(\theta_1+\theta_2)\right]\right\}+O(1),\\
&\tau_{12}(x_1,x_2;\lambda) = \tau_0\Gamma_2(\lambda)\frac{r}{\sqrt{(r_1r_2)}}\left\{4\sin\left[\theta-\frac{1}{2}(\theta_1+\theta_2)\right]\right.\\
&\qquad\left.-\frac{1-2\nu}{1-\nu}\frac{a^2\sin\theta}{r_1r_2}\cos\left[\frac{3}{2}(\theta_1+\theta_2)\right]\right\}+O(1),\\
&\tau_{21}(x_1,x_2;\lambda) = -\frac{\tau_0a^2(1-2\nu)\Gamma_2(\lambda)r}{(1-\nu)(r_1r_2)^{3/2}}\left\{\sin\theta\cos\left[\frac{3}{2}(\theta_1+\theta_2)\right]\right\}+O(1),\\
&\sigma_1(x_1,x_2;\lambda) = -\frac{\tau_0a^2\Gamma_1(\lambda)}{2(1-\nu)}\frac{1}{\sqrt{(r_1r_2)}}\sin\left[\frac{1}{2}(\theta_1+\theta_2)\right]+O(1),\\
&\sigma_2(x_1,x_2;\lambda) = \frac{\tau_0a^2\Gamma_1(\lambda)}{2(1-\nu)}\frac{1}{\sqrt{(r_1r_2)}}\cos\left[\frac{1}{2}(\theta_1+\theta_2)\right]+O(1).
\end{aligned}\right\} \tag{4.4}$$

We emphasize that the functions designated "$O(1)$"—though bounded in the closure of D^+—are by no means analytic at $r_\gamma = 0$.

It is apparent‡ from (4.3), (4.4) that

$$\mathring{u}_\alpha(x_1,x_2) = O(1), \qquad \mathring{\omega}(x_1,x_2) = O(r_\gamma^{-\frac{1}{2}}) \qquad \mathring{\tau}_{\alpha\beta}(x_1,x_2) = O(r_\gamma^{-\frac{1}{2}}) \tag{4.5}$$

as $r_\gamma \to 0$, whereas in this limit (for fixed $\lambda > 0$)

$$\left.\begin{aligned}
&u_\alpha(x_1,x_2;\lambda) = O(1), \qquad \omega(x_1,x_2;\lambda) = O(1),\\
&\tau_{\alpha\beta}(x_1,x_2;\lambda) = O(r_\gamma^{-\frac{1}{2}}), \qquad \sigma_\alpha(x_1,x_2;\lambda) = O(r_\gamma^{-\frac{1}{2}}).
\end{aligned}\right\} \tag{4.6}$$

Thus the displacements remain bounded at the endpoints of the crack in both the modified and the classical solution. On the other hand, the rotation field, which becomes infinite as $r_\gamma \to 0$ according to the conventional theory, remains finite when couple-stresses are taken into account. Further, the ordinary stress field grows beyond bounds as $r_\gamma \to 0$ in

† See also [1], where the same procedure is illustrated in detail.

‡ Observe that the functions within braces in (4.3), (4.4) remain bounded as $r_\gamma \to 0$ ($\gamma = 1, 2$).

both solutions and the order of these stress singularities is preserved in the departure from the classical theory. Finally, the order of the singularities at $r_\gamma = 0$ inherent in the couple-stress field is the same as that of the ordinary stress singularities. The latter conclusion is consistent with the behavior of ω exhibited in (4.6)† since, by virtue of (1.6) and (1.7), σ_α is proportional to $\omega_{,\alpha}$. The fact that $\tau_{\alpha\beta}$ and σ_α have singularities of equal order is, however, surprising in view of the second of (1.15): evidently, the higher-order singularities generated through the differentiation of $\tau_{\alpha\beta}$ cancel upon forming the linear combination of stress-derivatives appearing in the right-hand member of this compatibility relation.‡

It is important to observe that while the *order* of the singularities of $\tau_{\alpha\beta}$ in (4.4) and of $\mathring{\tau}_{\alpha\beta}$ in (4.3) is the same, i.e. $O(r_\gamma^{-1/2})$, the *detailed structure* of these singularities is different. Indeed, the singular terms exhibited in (4.4) involve ν, whereas $\mathring{\tau}_{\alpha\beta}$ is independent of Poisson's ratio.

We proceed now to the discussion of the numerical results obtained. The two quantities of primary physical interest are the transverse displacement u_2 at the center of the crack and the transverse normal stress τ_{22} along the extended crack-axis near the endpoints of the crack: the first supplies a measure of the deformations; the second is indicative of the most prominent stress-concentration effect.

From (2.12), (3.42), and (4.3) one finds by recourse to known Bessel integrals (Watson [6], p. 405) that

$$\frac{u_2(0,0;\lambda)}{\mathring{u}_2(0,0)} = \frac{1}{1-\nu}\left[\Gamma_2(\lambda)+\int_0^1(\sqrt{\xi})\Phi(\xi;\lambda)\,d\xi\right] \qquad (0<\lambda<\infty). \tag{4.7}$$

On the other hand, (4.3) and (4.4) furnish directly

$$\lim_{x_1\to a+}\left[\frac{\tau_{22}(x_1,0;\lambda)}{\mathring{\tau}_{22}(x_1,0)}\right] = \frac{3-2\nu}{1-\nu}\Gamma_2(\lambda) \qquad (0<\lambda<\infty). \tag{4.8}$$

It is clear from (2.5), (2.6) that formulas (4.7), (4.8) are also applicable to the problem of a crack in a transverse field of tension.

The numerical evaluation of (4.7), (4.8) was carried out on an IBM-7094 electronic computer for various values of the characteristic length-ratio $\lambda = l/a$ and for the values of Poisson's ratio $\nu = 0, \frac{1}{4}, \frac{1}{2}$. The required preliminary computations include the numerical solution of the triplet of independent Fredholm integral equations (3.37) for the unknown functions Φ_i and the subsequent determination of Γ_α and Φ from (3.38) and (3.39).

Figure 3 depicts the ratio of the transverse displacement at the center of the crack to the corresponding classical value as a function of $\lambda = l/a$ for the three values of Poisson's ratio at hand. Since the origin is a regular point,

$$\lim_{\lambda\to 0}\frac{u_2(0,0;\lambda)}{\mathring{u}_2(0,0)} = 1 \tag{4.9}$$

and thus the transition to the classical theory is continuous as far as the displacement under consideration is concerned. As is apparent, the opening of the crack at its midpoint diminishes monotonically in the departure from the classical theory, i.e. as λ increases,

† Actually, one finds that ω is not merely $O(1)$ but also $O(r_\gamma^{1/2})$ as $r_\gamma \to 0$.
‡ Cf. the related observations in [1].

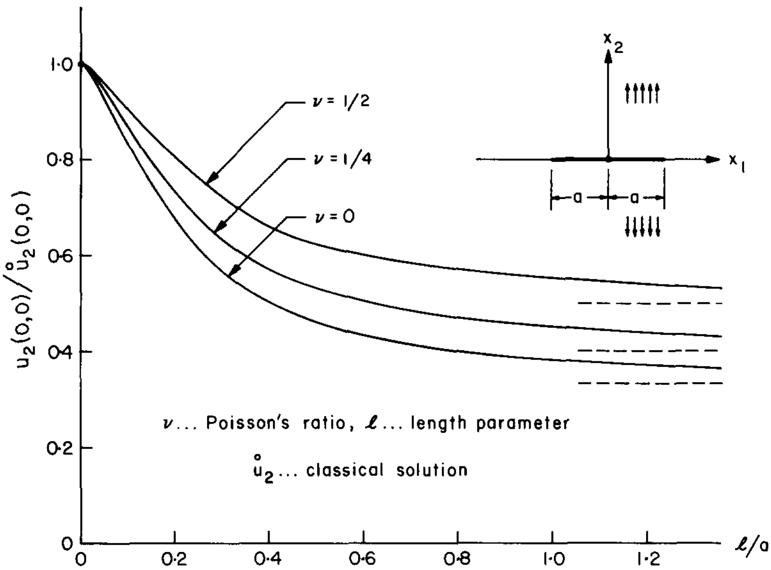

FIG. 3. Crack in a transverse field of tension. Transverse displacement u_2 at center of crack.

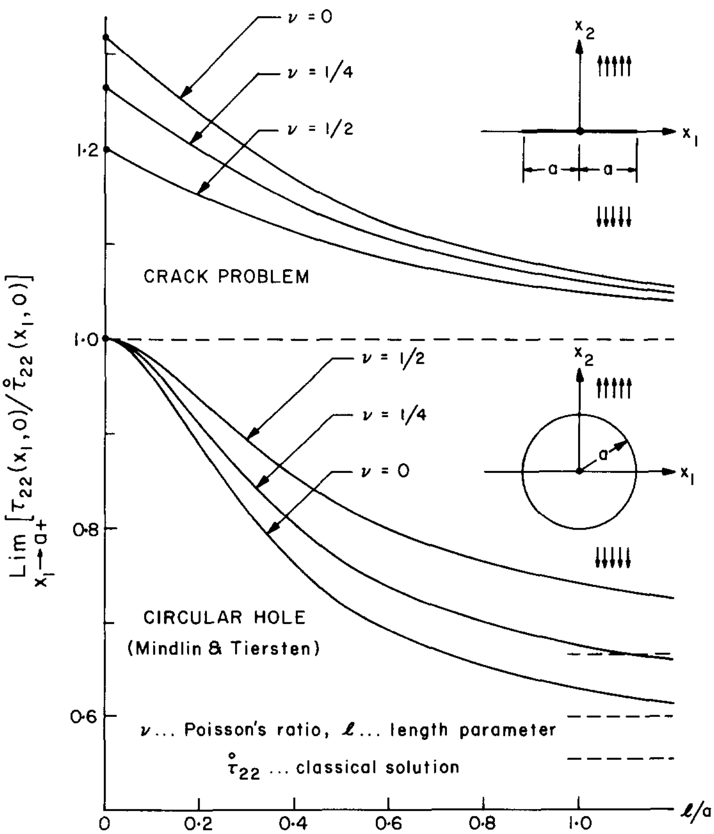

FIG. 4. Crack in a transverse field of tension. Transverse normal stress τ_{22} at root of crack.

this decrease being more pronounced at smaller values of Poisson's ratio. The asymptotic values of the displacement-ratio in the limit as $\lambda \to \infty$ are given by

$$\lim_{\lambda\to\infty} \frac{u_2(0,0;\lambda)}{\mathring{u}_2(0,0)} = \frac{1}{3-2\nu}. \tag{4.10}$$

To justify (4.10) refer to (4.7) and note with the aid of (3.30) to (3.35) that

$$\lim_{\lambda\to\infty} \Phi(\xi;\lambda) = 0 \qquad (0 \le \xi \le 1), \qquad \lim_{\lambda\to\infty} \Gamma_2(\lambda) = \frac{1-\nu}{3-2\nu}. \tag{4.11}$$

Figure 4 displays the dependence upon λ of the limit as $x_1 \to a+$ of $\tau_{22}(x_1,0;\lambda)/\mathring{\tau}_{22}(x_1,0)$, which remains finite† and characterizes the modification of the stress singularity when couple-stresses—or, equivalently, rotation gradients—are taken into account. On the same figure we have, for comparison purposes, superimposed the curves depicting the dependence on l/a of the analogous stress-ratio appropriate to the Mindlin–Tiersten [2]‡ solution for a circular hole of radius "a" in a transverse field of uni-axial tension.

It is seen from Fig. 4 that while the stress concentration at the hole is mitigated in the modified theory, the latter theory predicts an aggravation of the concentration of stress arising in the crack-problem. Indeed, the numerical results under discussion indicate that

$$\lim_{\lambda\to 0} \lim_{x_1\to a+} \left[\frac{\tau_{22}(x_1,0;\lambda)}{\mathring{\tau}_{22}(x_1,0)}\right] > 1, \tag{4.12}§$$

whereas evidently

$$\lim_{x_1\to a+} \lim_{\lambda\to 0} \left[\frac{\tau_{22}(x_1,0;\lambda)}{\mathring{\tau}_{22}(x_1,0)}\right] = 1. \tag{4.13}$$

Therefore the limiting stress-ratio plotted in Fig. 4 in the case of the crack problem exhibits a finite jump discontinuity at $l/a = 0$: the ratio of the modified to the classical transverse normal stress at the roots of the crack rises abruptly as l/a departs from zero; this ratio then declines monotonically with increasing values of l/a and tends to unity as $l/a \to \infty$ since

$$\lim_{\lambda\to\infty} \lim_{x_1\to a+} \left[\frac{\tau_{22}(x_1,0;\lambda)}{\mathring{\tau}_{22}(x_1,0)}\right] = 1 \tag{4.14}$$

according to (4.8) and the second of (4.11). The discontinuous behavior described above appears to be typical of the severe boundary-layer effects predicted by the couple-stress theory in singular stress-concentration problems.‖

It is easily found from the third of (4.3) that

$$\mathring{\omega}_{,2}(x_1,0) = 0 \quad (-a < x_1 < a). \tag{4.15}$$

Hence, according to the theorem established in Section 1, the *classical* displacement,

† See (4.8). Recall that τ_{22} and $\mathring{\tau}_{22}$ have singularities of the same order at the endpoints of the crack.

‡ See also [3].

§ The analytical evaluation of the iterated limit (4.12) appears to be exceedingly difficult.

‖ Cf. the analogous conclusions reached in [1].

rotation, and stress fields $\mathring{u}$, $\mathring{\omega}$, $\mathring{t}_{\alpha\beta}$ given in (4.3), supplemented by the couple-stress field

$$\sigma_\alpha(x_1, x_2) = 4\mu l^2 \mathring{\omega}_{,\alpha}, \tag{4.16}$$

conform to the *modified* plane-strain field equations on D and meet the boundary conditions (2.3), as well as the regularity conditions at infinity (2.4). Explicitly, (4.16) and (4.3) now furnish the couple-stresses

$$\left.\begin{aligned} \sigma_1(x_1, x_2) &= \frac{4a^4\tau_0(1-\nu)\lambda^2}{(r_1 r_2)^{3/2}} \sin\left[\frac{3}{2}(\theta_1+\theta_2)\right], \\ \sigma_2(x_1, x_2) &= -\frac{4a^4\tau_0(1-\nu)\lambda^2}{(r_1 r_2)^{3/2}} \cos\left[\frac{3}{2}(\theta_1+\theta_2)\right], \end{aligned}\right\} \tag{4.17}$$

whose singularities are $O(r_\gamma^{-\frac{3}{2}})$ as $r_\gamma \to 0$, in contrast to those inherent in the couple-stress field deduced earlier,† which are merely $O(r_\gamma^{-\frac{1}{2}})$.

The preceding, entirely elementary and closed, alternative "solution" to the problem of the pressurized crack, which exhibits no interaction between ordinary and couple-stresses, may safely be dismissed as a physically irrelevant pseudo-solution. This claim is further supported by the observation that the total strain energy associated with the conventional stress field appearing in (4.3) and the couple-stress field (4.17) is no longer finite, as is apparent from (1.11).

It should be emphasized in closing that the solution to the pressurized-crack problem deduced in Section 3 rests on the assumption (3.10) concerning the nature of the unknown singularities. This assumption, in turn, was motivated by the singular results in [1] for the problems of the concentrated load and of the discontinuous shear load, which were established through appropriate limit processes. A comparably satisfactory validation of the solution to the crack problem would require solving first the analogous problem for an elliptic hole and subsequently passing to the limit as the elliptic boundary degenerates into a straight-line segment. Unfortunately the problem of the elliptic hole within the couple-stress theory would appear to be one of prohibitive complexity.

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APPENDIX

We sketch here a proof of the order-of-magnitude estimates (4.1), which pertain to the limit of the solution to the crack problem as $\lambda = l/a \to 0$. To this end, as is apparent

† See the last of (4.6).

from (3.42), it suffices to show that

$$\left.\begin{aligned}[\Gamma_2(\lambda)-(1-\nu)]J_1(as)+\int_0^1 \sqrt{(\xi)}\Phi(\xi;\lambda)J_1(as\xi)\,d\xi &= o(1),\\ \Gamma_1(\lambda) = o(1) \quad \text{as} \quad \lambda\to 0 \qquad (0< as <\infty).\end{aligned}\right\} \tag{A1}$$

As a preliminary to our present objective we examine the behavior, as $\lambda \to 0$, of the auxiliary functions K and F_i entering the integral equation (3.30). From the last of (2.10), as well as (3.16), (3.31), and (3.32), follow the estimates—valid as $\lambda \to 0$:

$$\left.\begin{aligned}(3-2\nu)\lambda K(\xi,\eta;\lambda) &= L(\xi,\eta)+o(1) \qquad (0\le\xi\le 1, 0\le\eta\le 1, \xi\neq\eta),\\ (3-2\nu)\lambda F_2(\xi;\lambda) &= -L(\xi,1)+o(1) \qquad (0\le\xi<1),\\ (3-2\nu)\lambda F_3(\xi;\lambda) &= (1-\nu)L(\xi,1)+o(1) \qquad (0\le\xi<1),\end{aligned}\right\} \tag{A2}$$

where

$$L(\xi,\eta) = L(\eta,\xi) = \sqrt{(\xi\eta)}\int_0^\infty J_1(\xi s)J_1(\eta s)\,ds \qquad (0\le\xi\le 1, 0\le\eta\le 1, \xi\neq\eta). \tag{A3}$$

The function L becomes unbounded as $\xi\to\eta$. Indeed, an examination of (A3) reveals that in the limit as $\xi\to\eta$,

$$L(\xi,\eta) = -\frac{1}{\pi}\sqrt{\left(\frac{\eta}{\xi}\right)}\left[\log\left(1-\frac{\xi^2}{\eta^2}\right)+O(1)\right] \qquad (0<\xi<\eta\le 1). \tag{A4}$$

Also, by means of contour integration one deduces from (A3) the alternative representation

$$L(\xi,\eta) = \frac{2\sqrt{(\xi\eta)}}{\pi}\int_0^\infty I_1(\xi t)K_1(\eta t)\,dt \qquad (0\le\xi<\eta\le 1) \tag{A5}$$

in terms of the modified Bessel functions I_1 and K_1. This representation, in conjunction with (3.43) and the second of (3.32), enables one to infer that

$$\left.\begin{aligned}\lambda|K(\xi,\eta;\lambda)| &< 2L(\xi,\eta) \qquad (0\le\xi\le 1, 0\le\eta\le 1),\\ \lambda|F_2(\xi;\lambda)| &< 2L(\xi,1) \qquad (0\le\xi\le 1),\\ \lambda|F_3(\xi;\lambda)| &< L(\xi,1) \qquad (0\le\xi\le 1),\end{aligned}\right\} \tag{A6}$$

for every $\lambda > 0$. As for the behavior of F_1, we shall require merely the following integral property, which may be established on the basis of (3.43):

$$(3-2\nu)\lambda\int_0^1 h(\xi)F_1(\xi;\lambda)\,d\xi = \frac{h(1)}{2}\log\lambda+O(1) \quad \text{as} \quad \lambda\to 0 \tag{A7}$$

for every function h that is continuously differentiable on $[0, 1]$.

With a view toward confirming the second of (A1) multiply both sides of (3.30) by $\lambda h(\xi)$ and integrate the resulting identity with respect to ξ over the range $[0, 1]$ to obtain

$$\begin{aligned}\lambda\Gamma_1(\lambda)\int_0^1 h(\xi)F_1(\xi;\lambda)\,d\xi &= \lambda\int_0^1 h(\xi)\Phi(\xi;\lambda)\,d\xi+\lambda\int_0^1\int_0^1 h(\xi)\Phi(\eta;\lambda)K(\xi,\eta;\lambda)\,d\eta\,d\xi\\ &\quad -\lambda\Gamma_2(\lambda)\int_0^1 h(\xi)F_2(\xi;\lambda)\,d\xi-\lambda\int_0^1 h(\xi)F_3(\xi;\lambda)\,d\xi.\end{aligned} \tag{A8}$$

At this stage we assume

$$\Gamma_\alpha(\lambda) = O(1), \qquad \int_0^1 |\Phi(\eta;\lambda)| d\eta = O(1) \quad \text{as} \quad \lambda \to 0 \tag{A9}$$

and observe that, given $\varepsilon > 0$, one may evidently choose h in (A8) in such a way that it has the requisite smoothness and conforms to

$$\left.\begin{aligned} &0 \le h(\xi) \le 1 \qquad (0 \le \xi \le 1), \qquad h(1) = 1, \\ &\left|\int_0^1 h(\xi)L(\xi,\eta)\, d\xi\right| < \varepsilon \qquad (0 \le \eta \le 1). \end{aligned}\right\} \tag{A10}†$$

We show first that for such a choice of h the right-hand member in (A8) is $o(1)$, so that

$$\lambda\Gamma_1(\lambda) \int_0^1 h(\xi)F_1(\xi;\lambda)\, d\xi = o(1) \quad \text{as} \quad \lambda \to 0. \tag{A11}$$

To see this observe that the first term on the right-hand side of (A8) is $o(1)$ because of (A9) and the first of (A10). On the other hand, the remaining three terms may be made arbitrarily small in absolute value for sufficiently small positive values of λ by virtue of (A6), (A9), and (A10).

Now (A11), together with (A7) and (A10), are readily found to furnish

$$\Gamma_1(\lambda)\log(\lambda) = o(1) \quad \text{as} \quad \lambda \to 0, \tag{A12}$$

which in turn implies the second of (A1).

We turn next to the confirmation of the first of (A1). For this purpose we suppose Ψ to be a function bounded and integrable on $[0, 1]$ that satisfies the integral equation of Fredholm's first kind

$$\int_0^1 \Psi(\xi)L(\xi,\eta)\, d\xi - \sqrt{(\eta)}J_1(as\eta) = 0 \qquad (0 \le \eta \le 1). \tag{A13}$$

Multiplying both sides of (3.30) by $(3-2\nu)\lambda\Psi(\xi)$ and integrating the resulting identity with respect to ξ over the range $[0, 1]$ we are led to

$$\begin{aligned} &[\Gamma_2(\lambda)-(1-\nu)]J_1(as) + \int_0^1 (\sqrt{\xi})\Phi(\xi;\lambda)J_1(as\xi)\, d\xi = \\ &= (3-2\nu)\left[\lambda\Gamma_1(\lambda)\int_0^1 \Psi(\xi)F_1(\xi;\lambda)\, d\xi - \lambda\int_0^1 \Psi(\xi)\Phi(\xi;\lambda)\, d\xi\right] \\ &\quad - \int_0^1 \Phi(\eta;\lambda)\left[(3-2\nu)\lambda\int_0^1 \Psi(\xi)K(\xi,\eta;\lambda)\, d\xi - (\sqrt{\eta})J_1(as\eta)\right] d\eta \\ &\quad + \Gamma_2(\lambda)\left[(3-2\nu)\lambda\int_0^1 \Psi(\xi)F_2(\xi;\lambda)\, d\xi + J_1(as)\right] \\ &\quad + \left[(3-2\nu)\lambda\int_0^1 \Psi(\xi)F_3(\xi;\lambda)\, d\xi - (1-\nu)J_1(as)\right]. \end{aligned} \tag{A14}$$

It remains to be shown that the right-hand member of (A14) is $o(1)$ as $\lambda \to 0$. The first term on the right-hand side of (A14) tends to zero with λ since Ψ is bounded on $[0, 1]$, because F_1 is nonpositive in view of (3.43), and because of (A7), (A12). The second term is

† Note from (A4) that the singularity inherent in $L(\xi, \eta)$ is integrable.

$o(1)$ by virtue of (A9); the remaining terms approach zero with λ, as is seen with the aid of (A2), (A6), and (A13). This completes the proof of (A1) in its entirety.

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Résumé—Une solution à deformation plane est obtenue dans les limites de la théorie linéarisée couple-effort du comportement élastique pour le problème que présente une fissure limitée dans un champs transversal de tension uniforme et uni-axiale. Les phénoménes se produisant aux extrémités de la fissure sont étudiés en détail et les résultats, qui se rapportent aux considérations de fracture, sont comparés avec ceux analogues dans les corps élasto-statiques classiques.

Zusammenfassung—Eine Lösung des ebenen Verzerrungszustandes wird erhalten, im Rahmen der linearisierten Theorie für Elastizität unit Momentenspannungen, für das Problem eines begrenzten Risses im Transversalfeld einer gleichmässigen einachsigen Spannung. Die Bedingungen an den Rissenden werden genau untersucht und die Resultate werden mit den entsprechenden Werten der klassischen Elastizitätstheore verglichen.

Абстракт—Получено решение простой деформации в пределах линеаризированной теории парного напряжения эластичного поведения для проблемы, представленной конечной трещиной в поперечном поле равномерного одноосного напряжения. Особенности, возникающие наконцах трещины изучаются в деталях и результаты, которые подходят к рассмотрениям трещины, сравниваются с их взаимозаменяемой частьюв классической эласто-статике.